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THE ECHO SIGNAL OF A FINITE SPHERICAL PULSE FROM A FLUID FILLED SPHERICAL SHELL

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A method is proposed for determining the echo signal of a centrosymmetrical pressure pulse from a fluid filled elastic spherical shell immersed in an infinite fluid medium. The medium surrounding the shell and the fluid in it are considered to conform to the theory of perfect compressible fluid. Motion of the shell is defined in accordance with Timoshenko's linear theory of thin shells. The problem is solved with the use of Fourier's integral transformation in terms of time and Watson's transformation in terms of polar distance. Mechanisms of formation of the various echo signal components are described.

The determination of the reflected component of echo signal and of its components radiated by creeping and peripheral waves from empty spherical shells was considered in [1]. The effect of the echo signal of waves which pass either a fluid or an elastic cylinder was analyzed in [2, 3] by the method of Watson's transformation. Here, the echo signal from a fluid filled spherical shell is investigated, taking into account besides the indicated echo signal components also the effect of waves which propagate partly in the shell, as peripheral waves, and partly as waves that pass through the fluid filler.

1. The formal solution. With the use of conventional statement of the problem and expansion into series in Legendre polynomials in the space of Fourier transformation in terms of time, the echo signal of the probing pulse

$$p_i = A_0 l^{-1} f(\tau - l)\,[H(\tau - l) - H(\tau - l - \tau_p)] \tag{1.1}$$
$$l = (r_0^2 + r^2 - 2r_0 r \cos\vartheta)^{1/2}$$

reflected from a spherical shell filled with fluid may be represented in the form

$$p_e = A_0 \int_{-\infty}^{\infty} f^F \sum_{m=0}^{\infty} a_m x_m h_m^{(1)}(\omega r)\, P_m(\cos\vartheta)\, e^{-i\omega\tau}\, d\omega \tag{1.2}$$

$$f^F = \frac{1}{2\pi}\int_0^{\tau_p} f(\tau)\, e^{i\omega\tau}\, d\tau, \qquad a_m = i\omega r_0 (2m+1)\, h_m^{(1)}(\omega r_0) \tag{1.3}$$

$$x_m = -\frac{\omega A_{33} j_m(\omega) - (\alpha D + \alpha_0 D_0)\,\partial j_m(\omega)/\partial\omega}{\omega A_{33} h_m^{(1)}(\omega) - (\alpha D + \alpha_0 D_0)\,\partial h_m^{(1)}(\omega)/\partial\omega}$$

$$\alpha = \frac{h\rho_1 c_1^2}{R\rho c^2}, \quad \alpha_0 = \frac{\rho_0 c_0}{\rho c}, \quad D_0 = \frac{\omega A_{33} j_m(\beta_0\omega)}{\partial j_m(\beta_0\omega)/\partial(\beta_0\omega)}$$

$$\beta_0 = c c_0^{-1}, \quad D = \det\|a_{ij}\|; \quad i = 1, 2, 3;\ j = 1, 2, 3$$

In defining elastic waves in a shell in conformity with Timoshenko's theory [4] the elements of determinant D are of the form

$$a_{11} = 1 - \nu - M - \varkappa^2 + (1 + a^2)\beta^2\omega^2, \quad a_{12} = \varkappa^2 + 2a^2\beta^2\omega^2 \tag{1.4}$$
$$a_{13} = 1 + \nu + \varkappa^2, \; a_{21} = \varkappa^2 + 2a^2\beta^2\omega^2, \; a_{22} = a^2(1 - \nu - M + \beta^2\omega^2) - \varkappa^2, \; a_{23} = -\varkappa^2, \; a_{31} = (1 + \nu + \varkappa^2)M, \; a_{32} = -\varkappa^2 M$$
$$a_{33} = -\varkappa^2 M - 2(1 + \nu) + (1 + a^2)\beta^2\omega^2, \quad M = m(m + 1)$$
$$a = \frac{h}{\sqrt{12}R}, \quad \beta = \frac{c}{c_1}, \quad c_1 = \left[\frac{E}{(1 - \nu^2)\rho_1}\right]^{1/2}$$
$$\varkappa = k_T\left[\frac{1 - \nu}{2}\right]^{1/2}, \quad k_T = 0.912$$

The following notation is used in formulas (1.1) — (1.4): c and ρ are the speed of sound and the density of the medium surrounding the shell; c_0 and ρ_0 are the speed of sound and the density of the filler; r and ϑ are dimensionless (all length dimensions are in terms of the shell surface mean radius R) spherical coordinates whose pole is located at the center of the spherical shell ($\vartheta = 0$ for the position vector directed toward the center of the probing pulse source); τ is the dimensionless time ($\tau = ct/R$, t is the time and $t=0$ is the instant of switching on of the source); h is the shell thickness; E, ν and ρ_1 are, respectively, the modulus of elasticity, the Poisson's coefficient, and the density of the shell material; r_0 is the dimensionless distance between the center of the probing pulse and that of the shell; A_0 and f are, respectively, the constant which defines the amplitude of pressure variation in the probing pulse, and the law which defines the variation; τ_p is the dimensionless duration of the probing pulse; H is the Heaviside unit function; ω is the Fourier transformation parameter (frequency); P_m are Legendre polynomials, j_m and $h_m^{(1)}$ Bessel and Hankel functions of the first kind, and A_{33} is the corresponding algebraic minor of the determinant D.

To represent the echo signal in physically easily interpreted series in terms of peripheral and creeping waves we use Watson's transformation of the form [5]

$$\sum_{m=0}^{\infty} F(m, \omega) P_m(\cos\vartheta) = \sum_{n=0}^{\infty}\int_{\Gamma} F(\mu, \omega) e^{i\mu\pi(2n+1)} P_\mu[\cos(\pi - \vartheta)]\, d\mu \tag{1.5}$$

where the integration path Γ covers the positive part of the real axis in the complex plane μ in the clockwise direction.

Using (1.5) and the relation

$$j_\mu(x) = {}^1/_2\,[h_\mu^{(1)}(x) + h_\mu^{(2)}(x)] \tag{1.6}$$

we represent the echo signal (1.2) in the form

$$p_e = -\frac{A_0}{2}\int_{-\infty}^{\infty} f^F \sum_{n=0}^{\infty}\int_{\Gamma} a_\mu\left[1 + W\frac{h_\mu^{(2)}(\omega)}{h_\mu^{(1)}(\omega)}\right] S\, d\mu\, e^{-i\omega\tau} d\omega \tag{1.7}$$

$$S = e^{i\mu\pi(2n+1)} h_\mu^{(1)}(\omega r) P_\mu[\cos(\pi - \vartheta)], \quad W = F_0^{-1}F_1 - TUVF_0^{-2} \tag{1.8}$$
$$U = (1 - VF_2F_0^{-1})^{-1}, \quad V = h_\mu^{(1)}(\beta_0\omega)/h_\mu^{(2)}(\beta_0\omega)$$

$$F_0 = (\omega A_{33} - \alpha D E_1) G_2 - \alpha_0\omega A_{33}E_1$$
$$F_1 = (\omega A_{33} - \alpha D E_2) G_2 - \alpha_0\omega A_{33}E_2$$

$$F_2 = -(\omega A_{33} - \alpha DE_1) G_1 + \alpha_0 \omega A_{33} E_1$$
$$T = \alpha_0 \omega^2 A_{33}^2 (E_1 - E_2)(G_1 - G_2)$$
$$E_{1,2} = [\partial h^{(1,2)}(\omega) / \partial\omega] / h_\mu^{(1,2)}(\omega)$$
$$G_{1,2} = [\partial h_\mu^{(1,2)}(\beta_0\omega / \partial(\beta_0\omega)] / h_\mu^{(1,2)}(\beta_0\omega)$$

where a_μ, D and A_{33} are determined by formulas (1.3) and (1.4), respectively, with the substitution of μ for m.

The ratios $F_1 F_0^{-1}$, $F_2 F_0^{-1}$ and TF_0^{-2} in (1.8) have a definite physical meaning: the first two are the coefficients of internal and external reflection on the shell surface, while the third is the product of the coefficients of penetration from the surrounding medium to the filler and vice versa. It should be noted that part of the energy of penetrating waves is spent at each penetration through the shell on the generation of waves propagating in the shell, hence the relation between the related coefficients of penetration and reflection is more complex than in the case of the interface of two media.

By omitting unity appearing in brackets in (1.7) and representing function U in the form of series

$$U = \sum_{j=1}^{\infty} (VF_2F_0^{-1})^{j-1} \tag{1.9}$$

for the echo signal (1.7) we obtain

$$p_e = -\frac{A_0}{2} \int_{-\infty}^{\infty} f^F \sum_{n=0}^{\infty} \int_{\Gamma} \left(\frac{F_1}{F_0} - \sum_{j=1}^{\infty} T \frac{V^j F_2^{j-1}}{F_0^{j+1}} \right) K d\mu e^{-i\omega\tau} d\omega \tag{1.10}$$

$$K = a_\mu h_\mu^{(1)-1}(\omega)\, h_\mu^{(2)}(\omega)\, h_\mu^{(1)}(\omega r)\, e^{i\mu\pi(2n+1)} P_\mu[\cos(\pi - \vartheta)]$$

2. Inversion of transformations. Inversion of Watson's transformation is achieved approximately by the saddle-point method. For this the rapidly varying functions V and K in (1.10) are replaced by their approximate representations. Substituting the asymptotic formulas

$$h_\mu^{(1,2)}(\omega) \sim \omega^{-1}(1 + z^2)^{-1/4} \exp\{\pm i[\omega(\sqrt{1 - z^2} - z \arccos z) - \pi / 4]\} \tag{2.1}$$

$$P_\mu[\cos(\pi - \vartheta)] \sim \eta^{-1}(\exp\{i[\omega z(\pi - \vartheta) - \pi / 4]\} + \exp\{-i[\omega z(\pi - \vartheta) - \pi/4]\}) \tag{2.2}$$
$$\eta = [2\pi(\omega z - {}^1/_2)\sin\vartheta]^{1/2}, \quad z = (\mu + {}^1/_2) / \omega$$

into formulas for V and K in (1.8) and (1.10), we obtain

$$V = \exp\{2i\omega\beta_0 [(1 - z^2\beta_0^{-2})^{1/2} - z\beta_0^{-1}\arccos(z\beta_0^{-1})] - i\pi/2\} \tag{2.3}$$

$$K = 2zNr^{-1}\eta^{-1} \sum_{k=1}^{2} \exp\left\{i\omega[d_0 + z(g_0 + \vartheta_{nk})] + i\left[(3 - 2k)\frac{\pi}{4} - n\pi\right]\right\}$$

$$N = [(1 + z^2 r_0^{-2})(1 + z^2 r^{-2})]^{-1/4}, \quad d_0 = (r_0^2 - z^2)^{1/2} + (r^2 - z^2)^{1/2} - 2(1 - z^2)^{1/2}$$

$$g_0 = 2 \arccos z - \arccos (zr_0^{-1}) - \arccos (zr^{-1})$$
$$\vartheta_{n1} = \vartheta + 2n\pi, \quad \vartheta_{n2} = 2\pi - \vartheta + 2n\pi$$

Substituting (2.3) into (1.10) we obtain for the echo signal the expression of the form

$$p_e = \frac{A_0}{2r} \int_{-\infty}^{\infty} f^F \sum_{k=1}^{2} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} T_{jnk} e^{-i\omega\tau} d\omega \tag{2.4}$$

where

$$T_{0nk} = -2e^{i\overset{\circ}{\alpha}_{nk}} \int_{\Gamma_z} \omega z N \eta^{-1} F_0^{-1} F_1 \exp [i\omega\varphi_{nk}^0 (z)]\, dz \tag{2.5}$$

$$T_{jnk} = 2e^{i\alpha_{nk}^j} \int_{\Gamma_z} \omega z N \eta^{-1} T F_0^{-j-1} F_2^{j-1} \exp [i\omega\varphi_{nk}^j (z)]\, dz, \quad j = 1, 2, \ldots \tag{2.6}$$

$$\varphi_{nk}^j (z) = d_j (z) + z [g_j (z) + \vartheta_{nk}], \quad \alpha_{nk}^j = (3 - 2k)\pi / 4 - n\pi - j\pi / 2$$
$$d_j (z) = d_0 + 2j\beta_0 (1 - z^2\beta_0^{-2})^{1/2}, \quad g_j (z) = g_0 - 2j \arccos (z\beta_0^{-1}),$$
$$j = 0, 1, 2, \ldots$$

The integration path Γ_z is of the same form in the z-plane as that of Γ in the μ-plane.

The coordinates $z = z_{jnk}$ of saddle points of integrals (2.5) and (2.6) are determined by the equation

$$g_j (z) + \vartheta_{nk} = 0, \quad j = 0, 1, 2, \ldots$$

which has solutions $z = z_{jnk}$ only when the following conditions are, respectively, satisfied:

$$0 \leqslant \vartheta_{nk} \leqslant \arccos r_0^{-1} + \arccos r^{-1}, \quad j = 0 \tag{2.7}$$

$$\arccos (\beta_0 r_0^{-1}) + \arccos (\beta_0 r^{-1}) - 2\arccos \beta_0 \leqslant \vartheta_{nk} \leqslant j\pi, \quad \beta_0 \leqslant 1, \quad j = 1, 2, \ldots$$

$$\arccos r_0^{-1} + \arccos r^{-1} + 2j \arccos \beta_0^{-1} \leqslant \vartheta_{nk} \leqslant j\pi; \quad \beta_0 \geqslant 1, \ j = 1, 2, \ldots$$

The saddle points of integrals (2.5) and (2.6) lie on the real axis of the z-plane between 0 and 1, and the curves of steepest descent are perpendicular to the axis. To compute integrals (2.5) we first substitute path Γ' for Γ_z and, since in the second and fourth quarters there are no singular points, path Γ' is, in turn, replaced by the path consisting of sections Γ_∞' and Γ_{0nk} (Fig. 1, a). For integrals (2.6) path Γ_z is replaced by a path consisting of sections Γ_∞ and Γ_{jnk} (Fig. 1, b). The contribution of sections Γ_∞ and Γ_∞' of integration paths are disregarded.

It should be noted that in the computation of integrals (2.5) it is necessary to take into account besides the contribution of saddle points, also the contribution of poles of the first quarter of the z-plane, which lie to the left of saddle points, and in the computation of integrals (2.6) to consider the contribution of poles of the first quarter which lie to the right of the saddle point. When a saddle point is absent, i.e. condition (2.7) is violated, either on the left or right, it is necessary to investigate the contribution of which poles are to be taken into account in integrals (2.5) and (2.6) by determining whether condition (2.7) is violated on the left or right. If it is violated on the left, the contributions of all poles are disregarded, and if it is violated on the right, the contributions of all poles in the first quarter of the z-plane are to be taken into account.

For computing the contributions of saddle points and poles, whose coordinates $z = z_s$ are determined by solutions of the equation $F_0 = 0$, the integrals (2.5) and (2.6) can be represented in the form

$$T_{jnk} = \overset{\circ}{G}_{jnk} e^{i\omega d_j(z_{jnk})} + \sum_{s=1}^{\infty} G^s_{jnk} e^{i\omega\varphi^j_{nk}(z_s)}, \quad j = 0, 1, 2, \ldots \tag{2.8}$$

where

$$\overset{\circ}{G}_{0nk} = [F_0^{-1} F_1 N C_0 B]_{z=z_{0nk}} \tag{2.9}$$

$$G^s_{0nk} = \frac{4\pi i\omega z_s}{\eta}\left[\frac{F_1 N}{\partial F_0/\partial z}\right]_{z=z_s} e^{i\overset{\circ}{\alpha}_{nk}} H(\vartheta_{nk} - \vartheta_s)$$

$$\overset{\circ}{G}_{jnk} = [F_0^{-j-1} F_2^{j-1}\, TBC_jN]_{z=z_{jnk}} e^{i(\alpha^j_{nk} - \pi/4)} \tag{2.10}$$

$$G^s_{jnk} = \frac{4\pi i\omega}{e^{i\omega\varphi^j_{nk}(z_s)}}\operatorname{Res}\left[\frac{zTF_2^{j-1}N}{\eta F_0^{j+1}} e^{i\omega\varphi^j_{nk}(z)}\right]_{z=z_s}, e^{i\alpha^j_{nk}} H(\vartheta_{nk} - \vartheta_s)$$

$$j = 1, 2, \ldots$$

$$G_j = \{1 - {}^1/_2(1 - z^2)^{1/2}[(r_0^2 - z^2)^{-1/2} + (r^2 - z^2)^{-1/2} + 2j\beta_0^{-1}(1 - z^2\beta_0^{-2})^{-1/2}]\}^{-1/2}$$

$$B = 2z\eta^{-1}[\pi\omega(1 - z^2)^{1/2}]^{1/2}, \quad \vartheta_s = -\operatorname{Re} g_j(z_s), \quad j = 0, 1, 2, \ldots$$

In the absence of saddle points functions $\overset{\circ}{G}_{jnk} \equiv 0$ for any j, and in formulas for G^s_{jnk} in (2.9) and (2.10) we have $\vartheta_s = j\pi$.

Inversion of the Fourier transformation in terms of time is carried out by the method proposed in [1, 6].

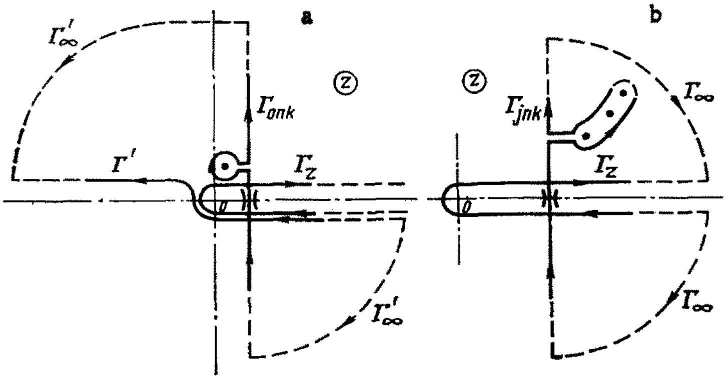

Fig. 1

3. Analysis of the results obtained. Functions T_{jnk} in (2.4) have the meaning of components of the stationary echo, i.e. of components of the echo of the stationary probing wave

$$p_i = 2rl^{-1}e^{i\omega l}$$

from the fluid filled spherical shell. Functions $\overset{\circ}{G}_{jnk}$ and G^s_{jnk} in (2.8) define the complex amplitudes, and d_j and φ^j_{nk} the times of arrival of individual components of the sationary echo. The first term in (2.8) determines the stationary echo components induced by waves which pass through the shell and filler and reflected several times from the inner shell surface during their propagation through the filler and, then, again passing through the shell into the surrounding medium. The terms under the sign of summation by s in (2.8) define the stationary echo components induced by waves which propagate over a part of their path in the shell and alongside it in the form of peripheral and creeping waves, and pass through the filler over another part of the path.

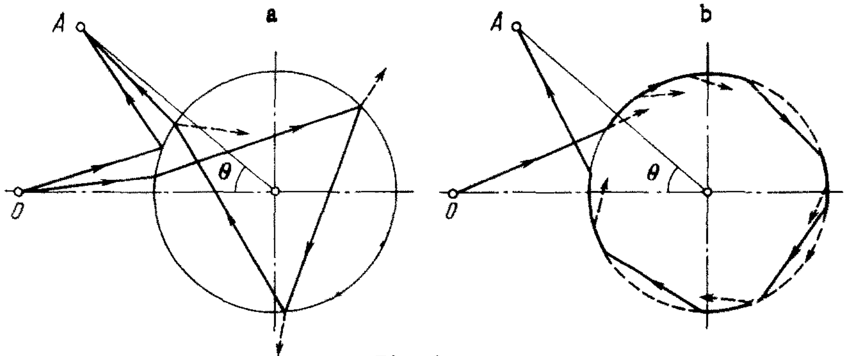

Fig. 2

The subscript j indicates in all terms the number of times a wave passes through the filler; terms with $j = 0$ define the components of the echo signal in the case of an empty spherical shell. The number n indicates how many complete cycles are completed by the waves in the shell or the filler prior to departure to the observation point, while number k shows whether such waves are propagated clockwise ($k = 1$) or counterclockwise ($k = 2$) . The number s in (2.8) is the number of the pole, i.e. the number of modes of peripheral or creeping waves. In specific computations it is often sufficient to take into consideration only the contributions of the zero-moment ($s = 1$) and flexural ($s = 2$) modes of peripheral waves.

The paths from the probing pulse source O to point A of observation of waves that pass through and are reflected from the shell inner surface are shown in Fig. 2, a for $n = 0$, $k = 1$, $j = 0$ and $k = 1$, $n = 1$, $j = 3$. One of the infinitely great numbers of possible paths of one mode of peripheral waves appear in Fig. 2, b for $n = 1$, $k = 1$ and $j = 3$.

It should be noted that the asymptotic representation (2.2) is only valid when

$$0 < \varepsilon \leqslant \vartheta \leqslant \pi - \varepsilon, \quad |\mu| \gg \varepsilon^{-1} \tag{3.1}$$

These conditions are not always satisfied. Let us examine the possiblity of extending the applicability region beyond the range indicated in (3.1). The behavior of function $P_\mu(\cos\vartheta)$ implies that for $|\mu| \leqslant \varepsilon^{-1}$ the asymptotic formula (2.2) can also be applied in the case of small $\sin\vartheta$, if function η is approximated by a formula that is more accurate in this region. Such formula may be of the form

$$\eta = \sqrt{2}\,(1 + \chi^{-1}\sin\vartheta), \quad \chi = 4\pi^{-1}(\omega z - {}^1/_2)^{-1} \tag{3.2}$$

It is applicable when $\sin\vartheta < |\chi|$. If $\sin\vartheta \geqslant |\chi|$, function η can be determined by formula (2.2).

For determining the echo signal in sectors in which $\sin\vartheta$ and z_{jnk} are small, i.e.

when the two conditions (3. 1) are violated, components of the echo signal image can also be determined by formulas (2. 8) — (2. 10), with function B appearing in these determined by formula

$$B = [2z (1 - z^2)^{1/2} \sin^{-1} \vartheta]^{1/2} \tag{3.3}$$

although the application of formula (3. 3) is justified only for considerable values of z and $\sin \vartheta$.

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ON A PARTICULAR CLASS OF SOLUTIONS OF TRIPLE INTEGRAL EQUATIONS

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A new class of solutions of triple integral equations is proposed. A number of boundary value problems of the elasticity theory with mixed boundary conditions (problems of contact, cracks, etc.) can be reduced to this class.

1. Let us consider triple integral equations of the form

$$\int_0^\infty \Phi(\xi) J_\nu(\xi x)\, d\xi = G_1(x) \quad (0 < x < a) \tag{1.1}$$

$$\int_0^\infty \xi^{-2\alpha} \Phi(\xi) J_\nu(\xi x)\, d\xi = F_2(x) \quad (a < x < b)$$

$$\int_0^\infty \Phi(\xi) J_\nu(\xi x)\, d\xi = G_3(x) \quad (b < x < \infty)$$

where functions G_1, F_2 and G_3 are assumed known, Φ is the unknown function, and $J_\nu(x)$ is a Bessel function of the first kind.